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Note(s)

Moments and cumulants for the complex Wishart

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ABSTRACT

We summarize the main results known for the complex normal and complex Wishart, then give the cumulants of the central and noncentral complex Wishart. Their moments are expressed explicitly in terms of multivariate Bell polynomials, believed to be used here for the first time. Multivariate Bell polynomials are easily written down from their univariate forms, which are widely accessible in most computer algebra packages. This is shown to be the natural way of obtaining the moments for any sum of independent and identically distributed (i.i.d.) random variables. An extension is given to the weighted complex Wishart.

1. Introduction

The complex Wishart distribution arises from the complex normal distribution just like the usual Wishart distribution arises from the multivariate normal distribution. The complex normal distribution was introduced by the New Zealander Wooding [33]. The complex Wishart distribution derived from it is due to Goodman [5,6] and Srivastava [27]. The mathematical properties of the complex Wishart distribution, including its eigenvalues, determinant, percentile points and characterizations, have been studied by many authors. For distributions related to the eigenvalues of central and noncentral complex Wishart distributions, see [22,12–14,26]. These results have applications to signal processing: for example, to obtain closed-form expressions for the outage probability of multiple input multiple output (MIMO) systems employing maximal ratio combining and operating over Rician-fading channels [14]. For distributions of the determinant of central and noncentral complex Wishart distributions, see [6,10,30]. These results have applications for the study of the capacity of wireless systems. For percentile points of individual eigenvalues of complex Wishart distributions, see [16,24]. Gupta and Kabe [9] provided a characterization of the complex Wishart distribution that generalizes the well-known result: if X and Y are independent positive random variables with XY and $(1-X)Y$ independent, then Y must be gamma and X beta.

Complex Wishart distributions have received applications in several areas. They have immediate applications in image analysis and related areas: when working with multi-look fully polarimetric synthetic aperture radar (SAR) data an appropriate way of representing the back-scattered signal consists of the so-called covariance matrix. For each pixel this is a 3×3 Hermitian, positive definite matrix which is known to follow a complex Wishart distribution, see [18,15,17,20,3,21,11]. The use of complex Wishart in this context has led to many benefits: for example, for unsupervised classification of terrain types and man-made objects using polarimetric SAR data [18,17] and for segmentation, change detection and edge detection in polarimetric SAR data [3,21]. Goodman [5], Goodman and Dubman [7] and Shaman [25] described applications of complex Wishart for the construction of spectral estimates in time series analysis. For instance, Shaman [25] used the

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complex Wishart to represent the distribution of an average of adjacent periodogram values. Forrester and Hughes [4] described an application to conductance in a mesoscopic wire (physics).

The moments of complex Wishart distributions are needed to study asymptotic properties of parameter estimates and to quantify performance in the applications mentioned above (for example, performance of signal processing algorithms). However, the known work on moments of complex Wishart distributions is either too limited or not too accessible for practical use. Shaman [25] gives the first and second moments of the inverted complex Wishart distribution. Tague and Caldwell [29] derive expressions for $E(WAW)$ and $E(W^{-1}AW^{-1})$ when W is a complex Wishart matrix and A a constant matrix. Sultan and Tracy [28] find the first four moments of the central and noncentral complex Wishart distributions by differentiating their characteristic functions using a matrix differential operator. Maiwald and Kraus [20] obtain the first four moments of the central complex Wishart distribution and also providing some approximate formulas. Lu and Richards [19] discuss obtaining moments of complex Wishart distributions by using MacMahon's master theorem and representation theory, but no explicit expressions are given. Graczyk et al. [8] find all moments of the complex Wishart distribution and its inverse using a technique of irreducible characters of a symmetric group. Most recently, Capitaine and Casalis [1] have considered generalized moments of $(W_1 + W_2)^{-1/2} W_1 (W_1 + W_2)^{-1/2}$ when W_1 and W_2 are independent complex Wishart matrices.

Graczyk et al. [8] is the only paper in the literature giving moments of all orders for the complex Wishart. However, what is given in Theorem 2 of [8], their main result, are generating functions for moments in terms of arbitrary complex matrices. It is, in fact, not a trivial step to obtain moments from Theorem 2. Furthermore, the derivations in [8] are too complicated and resort to concepts in group theory, as mentioned above. These derivations and the results themselves may not be understood by most practitioners, especially engineers. So, the practical value of the results in [8] is small.

This note derives for the first time explicit and accessible expressions for moments and cumulants of central and noncentral complex Wishart distributions. The moments are expressed explicitly in terms of multivariate Bell polynomials as a natural way of expressing multivariate moments in terms of multivariate cumulants. We believe multivariate Bell polynomials have not been used in the literature before. They are easily written down from their univariate forms. In-built routines for univariate Bell polynomials are available in most computer algebra packages, for example, see BellB in Mathematica. The given expressions for moments and cumulants hold for all possible values of the *degree of freedom* parameter (denoted by N in Section 2), integer or non-integer. Also the derivations presented are simple and natural. They do not resort to complicated concepts like group theory.

The results of this note are organized as follows. Section 2 gives two definitions of the complex normal. Section 3 summarizes relevant main results for the complex normal and complex Wishart. Section 4 gives the cumulants of the complex Wishart for both the central and noncentral cases, derived here for the first time. Section 5 gives their moments and Section 6 gives an extension to the scalar-weighted complex Wishart. To the best of our knowledge, the results presented in Sections 4–6 are new and original.

The following notation is used throughout this note. For z a complex vector, its transpose and complex conjugate are z^T and $\bar{z}$, respectively. Also, $z^+ = \bar{z}^T$.

2. The complex normal

Set $j = \sqrt{-1}$. A random complex M-vector Z with zero mean is said to be *complex normal*, written $Z \sim \mathcal{CN}_M(0, V)$ with *complex covariance* $V = EZZ^+$ in $\mathcal{C}^{M\times M}$ if its real and imaginary parts are jointly normal and its "real covariance" vanishes: $EZZ^T = 0$ in $\mathcal{C}^{M\times M}$. Wooding [33] gave the equivalent definition: Z is said to be complex normal, if $Z = X + jY$ and

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}_{2M}\left(0, \begin{pmatrix} A & B \\ -B & A \end{pmatrix}\right),$$

with A and B both $M \times M$. This is sometimes referred to as the *circular complex normal* to distinguish it from the case of arbitrary *covariance* $\binom{X}{Y}$. Note that

$$V = EZZ^+ = 2\,(A - jB)$$

and

$$\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \det^2(A - jB) = 2^{-2M}\det^2 V.$$

The simplest and most important example of a complex normal is $X + jY$, where X and Y are i.i.d. normal vectors. For $1 \le a \le M$, $EX_aY_a = 0$. So, if $M = 1$ then X and Y are i.i.d.

3. Main results for the complex normal and Wishart

In this section we summarize the important properties known for the complex normal and Wishart: the density of the complex normal, the moment generating functions for both, and the moments of the central complex Wishart.

Wooding [33] showed that for V of full rank (i.e. with minimum value positive), Z has density

$$f_Z(z) = \pi^{-M} (\det V)^{-1} \exp\left(-z^+ V^{-1} z\right)$$

for z in $\mathcal{C}^M$. This form of density allows us to work in $\mathcal{C}^M$ rather than in $\mathcal{R}^{2M}$, thus halving the dimensionality.

For r, s in $\mathcal{C}^M$ with transposes r', s' if $t = r + js$, $T = r'X + s'Y = (t^+Z + Z^+t)/2$, then by (20) of [33], $E\exp(T) = \exp(t^+Vt)/4$. We prefer the form

$$E\exp\left(t^+Z + Z^+t\right) = \exp\left(t^+Vt\right). \tag{1}$$

We write $H = Z + \mu \sim \mathcal{CN}_M(\mu, V)$. So, $EHH^+ = \mu\mu^+ + V$. Turin [31] showed that for $T^+ = T$ in $\mathcal{C}^{M\times M}$ and t in $\mathcal{C}$,

$$E\exp\left\{H^+THt\right\} = \det(I_M - tTV)^{-1}\exp(\gamma), \tag{2}$$

where the "noncentrality parameter" γ is given by $\gamma = \mu^+V^{-1}[(I - tVT)^{-1} - I]\mu$. Although not stated, this requires that $\lambda_1 \mathcal{R}e(t) < 1$, where λ_1 is the maximum eigenvalue of $V^{1/2}TV^{1/2}$. We give the alternative forms for γ : $\gamma = t\mu^+(T^{-1} - tV)^{-1}\mu = (I - tTV)^{-1}tT$.

For $H_1, \ldots, H_N$ with independent $\mathcal{CN}_M(\mu, V)$ the *noncentral complex Wishart* is defined as

$$\widetilde{W}_N = W_N(\mu, V) = \sum_{n=1}^{N} H_n H_n^+. \tag{3}$$

Here, N is the degree of freedom parameter. From (2), its moment generating function is given by

$$E\exp\operatorname{trace}\left(T\widetilde{W}_N\right) = \det(I_M - tTV)^{-N}\exp(N\gamma). \tag{4}$$

This was noted by Goodman [5,6] (with $t = 1$) for the *central* complex Wishart,

$$W_N = W_N(0, V) = \sum_{n=1}^{N} Z_n Z_n^+, \tag{5}$$

where $Z_1, \ldots, Z_N$ with independent $\mathcal{CN}_M(0, V)$. That paper also gives parallels to real theory for complex multiple coherence, correlation and conditional coherence. From a version of (1), Reed [23] (using the notation H for $V'/2$) proved

$$EZ_{a_1}\cdots Z_{a_r}\overline{Z}_{b_1}\cdots\overline{Z}_{b_s} = \delta_{rs}\sum_{p_1\cdots p_r}^{r!} V_{a_1p_1}\cdots V_{a_rp_r}$$

summed over all $r!$ permutations $p_1\cdots p_r$ of $b_1\cdots b_r$, where $\delta_{rs} = 1$ or 0 for $r = s$ or $r \neq s$. (His assumption of stationarity is not needed.)

For the complex normal, the gamma plays the role that the chi-square does for the real normal. For example, if $V = I_M$ then $|Z|^2 = \chi^2_{2M}/2 = G_M$, where G_M is a gamma random variable with mean M and density $x^{M-1}\exp(-x)/(M-1)!$ on $(0, \infty)$.

Note that although N is an integer in (3) and (5) the subsequent results on cumulants and moments of the complex Wishart also hold for all possible non-integer values of N. This follows by defining the distribution in terms of the moment generating function in (4).

4. Cumulants for the complex Wishart

Let us define the rth order cumulants of a random vector X in $\mathcal{C}^M$, $\kappa^{a_1\cdots a_r} = \kappa(X_{a_1}, \ldots, X_{a_r})$, for $1 \leq a_i \leq M$ exactly as for a random vector in $\mathcal{R}^M$. That is, for t in $\mathcal{C}^M$,

$$\ln E\exp\left(t^T X\right) = \sum_{r=1}^{\infty} \kappa^{a_1\cdots a_r} t_{a_1}\cdots t_{a_r}/r!,$$

where we use the tensor summation convention of implicit summation of repeated pairs $a_1, \ldots, a_r$ over their range $1, 2, \ldots, M$. For example, $\kappa(X_1, X_2) = EX_1X_2 - (EX_1)(EX_2)$ and not $EX_1\overline{X}_2 - (EX_1)(E\overline{X}_2)$. Repeatedly differentiating the logarithm of (3) with $N = 1$, $\mu = 0$, one obtains the following.

Theorem 4.1. *For $Z \sim \mathcal{CN}_M(0, V)$,*

$$\kappa\left(Z_{a_1}\overline{Z}_{b_1}, \ldots, Z_{a_r}\overline{Z}_{b_r}\right) = \sum_{C(p_1\cdots p_r)}^{(r-1)!} V_{a_1p_1}\cdots V_{a_rp_r} = k_{1\cdots r}, \tag{6}$$

say, summed over all $(r-1)!$ permutations $p_1\cdots p_r$ of $b_1\cdots b_r$ giving connected expressions. By connected we disqualify breaking $1\cdots r$ into two or more groups. For example, $V_{a_1b_1}V_{a_2b_2}$ breaks 12 into 1 and 2; $V_{a_1b_2}\,V_{a_2b_1}\,V_{a_3b_4}\,V_{a_4b_3}$ breaks 1234 into 12 and 34.

Theorem 4.2. *For $H \sim \mathcal{CN}_M(\mu, V)$,*

$$\kappa\left(H_{a_1}\overline{H}_{b_1}, \ldots, H_{a_r}\overline{H}_{b_r}\right) = k_{1\cdots r} + \gamma_{1\cdots r} = \widetilde{k}_{1\cdots r}, \tag{7}$$

say, where

$$\gamma_{1\cdots r} = \sum_{1\cdots r}^{r!} \mu_{a_1} v_{12} v_{23} \cdots v_{r-1,r} \overline{\mu}_{b_r},$$

$v_{rs} = V_{a_r b_s}$, and $\sum_{1\cdots r}^{r!}$ sums over all r permutations of suffixes $1 \cdots r$.

Proof. Set $A = A(T) = L^{-1}T, L = I - TV, \gamma(T) = \{\gamma \text{ of } (2)\}_{t=1} = \mu^+ A\mu$. For a complex matrix function $f(T)$, set $f_{\cdot 1\cdots r} = \partial_1 \cdots \partial_r f(T)$, where $\partial_i = \partial/\partial T_{b_i a_i}$. Set $B_r = -L_{\cdot r}$, the matrix with (i,j)th element $\delta_{ib_r} V_{ja_r}$. So, $(L^{-1})_{\cdot r} = L^{-1} B_r L^{-1}$ and $A_{\cdot 1} = L^{-1} B_1 L^{-1} T + L^{-1} T_{\cdot 1}$, $T_{\cdot r}$ is the matrix with (i,j)th element $\delta_{ib_r}\delta_{ja_r}$. Similarly,

$$A_{\cdot 1\cdots r} = \sum_{1\cdots r}^{r!} L^{-1} B_1 L^{-1} B_2 \cdots L^{-1}\left(B_r L^{-1} T + T_{\cdot r}\right) = \sum_{1\cdots r}^{r!} B_1 B_2 \cdots B_{r-1} T_{\cdot r}$$

if $T = 0$ and so $\gamma_{\cdot 1\cdots r}(0) = \gamma_{1\cdots r}$. The theorem now follows from the previous theorem by taking $\partial_1 \cdots \partial_r$ of the logarithm of (2) then putting $T = 0$. □

From the previous theorem it follows that the mean of the noncentral complex Wishart is $E\widetilde{W}_N = N(\mu\mu^+ + V)$ and for $r \geq 2$ its rth order cumulants are given by $\kappa(\widetilde{W}_{Na_1b_1}, \ldots, \widetilde{W}_{Na_rb_r}) = N\widetilde{k}_{1\cdots r}$.

5. Moments for the complex Wishart

We begin by expressing the moments of a sum of arbitrary i.i.d. random variables in terms of cumulants. We do this for scalars then vectors then matrices. Suppose $S = \sum_{n=1}^N X_n$ where X_n are independently-distributed as X in $\mathcal{C}$ with cumulants $\{\kappa_r\}$ defined by

$$\ln E \exp(tX) = \sum_{r=1}^{\infty} t^r \kappa_r / r! = K(t),$$

say, for t in $\mathcal{C}$. So,

$$E \exp(tS) = \exp\{NK(t)\} = \sum_{i=0}^{\infty} N^i (K(t))^i / i!.$$

For $\kappa = (\kappa_1, \kappa_2, \ldots)$ *any* sequence from $\mathcal{C}$, the *partial exponential Bell polynomial* $B_{ri} = B_{ri}(\kappa)$ is defined by

$$\left(\sum_{r=1}^{\infty} t^r \kappa_r / r!\right)^i \Bigg/ i! = \sum_{r=i}^{\infty} B_{ri} t^r / r!$$

for $r \geq i \geq 0$. They may be expressed in terms of the partition function. They are given in the table on page 30 of [2] up to $r = 12$. Since $B_{r0} = 0$ for $r \neq 0$, this gives the noncentral moment

$$ES^r = \sum_{i=1}^{r} N^i B_{ri} \tag{8}$$

for $r \geq 1$. Putting $\kappa_1 = 0$ gives the central moment

$$\mu_r(S) = E(S - ES)^r = \sum_{1 \leq i \leq r/2}^{r} N^i B_{ri0}, \tag{9}$$

where $B_{ri0} = B_{ri}|_{\kappa_1=0}$, since $B_{ri0} = 0$ for $i > r/2$. In fact, by (3.4) of [32], $B_{ri0} = (r)_i B_{r-i,i}(y)$ for $y_i = \kappa_{i+1}/(i+1)$, where $(r)_i = r(r-1)\cdots(r-i+1)$. For X in $\mathcal{C}^M$ and $a_1, \ldots, a_r = 1, 2, \ldots, M$, (8) and (9) become

$$ES_{a_1} \cdots S_{a_r} = \sum_{i=1}^{r} N^i B_i^{a_1\cdots a_r} \tag{10}$$

and

$$\begin{aligned} \mu\left(S_{a_1}, \ldots, S_{a_r}\right) &= E\left(X_{a_1} - EX_{a_1}\right)\cdots\left(X_{a_r} - EX_{a_r}\right) \\ &= \sum_{1 \leq i \leq r/2} N^i B_{i0}^{a_1\cdots a_r}, \end{aligned} \tag{11}$$

respectively, where $B_{i0}^{a_1\cdots a_r} = B_i^{a_1\cdots a_r}$ at $\kappa^1 = \cdots = \kappa^M = 0$ and $B_i^{a_1\cdots a_r} = B_i^{a_1\cdots a_r}(\kappa)$ is the *vector partial exponential Bell polynomial* defined by

$$\left(\sum_{r=1}^{\infty} \kappa^{a_1\cdots a_r} t_{a_1}\cdots t_{a_r}/r!\right)^i \Big/ i! = \sum_{r=i}^{\infty} B_i^{a_1\cdots a_r} t_{a_1}\cdots t_{a_r}/r!$$

for $r \geq i \geq 0$, t in $\mathcal{C}$ and $\kappa = \{\kappa^{a_1\cdots a_r}, r \geq 1, a_1, \ldots, a_r = 1, 2, \ldots, M\}$ *any* sequence of elements from $\mathcal{C}$. Again summation of repeated pairs $a_1, \ldots, a_r$ over their range is implicit. For X in $\mathcal{C}^{p\times q}$, one just replaces each suffix a by a double suffix ab:

$$ES_{a_1b_1}\cdots S_{a_rb_r} = \sum_{i=1}^{r} N^i B_i^{a_1b_1\cdots a_rb_r} \tag{12}$$

and

$$\mu\left(S_{a_1b_1}, \ldots, S_{a_rb_r}\right) = \sum_{1\leq i\leq r/2} N^i B_{i0}^{a_1b_1\cdots a_rb_r}, \tag{13}$$

where $B_i^{a_1b_1\cdots a_rb_r}$ is $B_i^{a_1\cdots a_r}$ with a_i replaced by (a_i, b_i) and similarly for $B_{i0}^{a_1b_1\cdots a_rb_r}$, these being given in terms of the rth order cumulants of X, $\kappa^{a_1b_1\cdots a_rb_r} = \kappa(X_{a_1b_1}, \ldots, X_{a_rb_r})$ for $1 \leq a_i \leq p$ and $1 \leq b_i \leq q$, $i = 1, 2, \ldots, r$. We call the $B_i^{a_1\cdots a_r}$ and $B_i^{a_1b_1\cdots a_rb_r}$ the *vector and matrix exponential Bell polynomials*. (Versions of them also arise naturally in the expression for the derivatives of a function of a vector or matrix function of a vector or matrix. See page 137 of [2] for the case of a function of a scalar function of a scalar.)

We can now give the noncentral and central moments of the complex Wishart $\widetilde{W}_N$ of (3): they are given by (12), (13) with $p = q = M$ and cumulants $\kappa^{a_1b_1\cdots a_rb_r}$ given by our main theorem (6) and (7) above for the central and noncentral cases.

The basic expressions for noncentral and central moments in terms of cumulants for a scalar/vector/matrix random variable are (8)–(13) at $N = 1$. Converse expressions for cumulants in terms of noncentral and central moments may be written down in the same manner. See [32] for the scalar case.

These results extend easily to sums of independent random variables $X_1, \ldots, X_N$ with different cumulants: one just replaces their rth order cumulants by their average, as illustrated in the next section.

6. The weighted complex Wishart

We now give a weighted version of the above results. Suppose $S_P = \sum_{n=1}^{N} P_n X_n$, where $\{P_n\}$ are constants in $\mathcal{C}$ and $\{X_n\}$ are i.i.d. as X in $\mathcal{C}$ with cumulants $\{\kappa_r\}$. Its cumulant generating function is

$$K_{S_P}(t) = \sum_{n=1}^{N} K_X(P_n t) = N\sum_{r=1}^{\infty} \alpha_r t^r/r! = NK(t)$$

for $\alpha_r = P_{rN}\kappa_r$ and $P_{rN} = N^{-1}\sum_{n=1}^{N} P_n^r$. So, for $r \geq 1$,

$$ES_P^r = \sum_{i=1}^{r} N^i B_{ri}(\alpha).$$

Similarly, for X in $\mathcal{C}^{p\times q}$, the noncentral and central moments of S_P are given by (12)–(13) with $\kappa^{a_1b_1\cdots a_rb_r}$ multiplied by P_{rN}.

We define the *weighted complex Wishart* as S_P with $X = ZZ^T$ for $Z \sim \mathcal{CN}_M(0, V)$ in the central case or with $X = HH^T$ for $H \sim \mathcal{CN}_M(\mu, V)$ in the noncentral case. So, its noncentral and central moments are given by (12)–(13) with $\kappa^{a_1b_1\cdots a_rb_r} = k_{1\ldots r}$ or $\widetilde{k}_{1\ldots r}$ multiplied by P_{rN}.

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